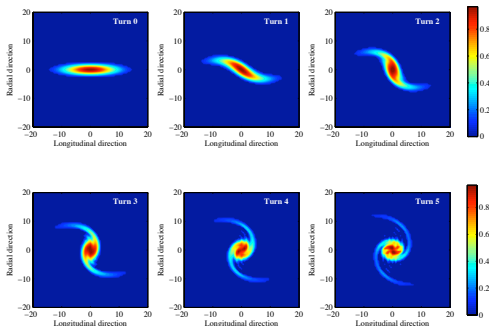


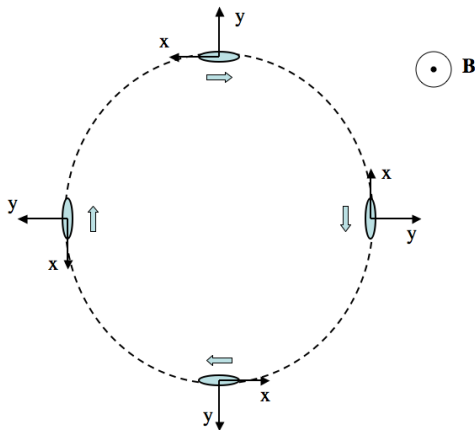
# Space charge in fixed-field rings



**Antoine Cerfon, Courant Institute NYU**  
with J. Guadagni, O. Bühler (Courant Institute NYU)  
J.P. Freidberg (MIT), and F.I. Parra (Oxford)

# MOVING FRAME

- The analysis is simpler and more elegant in the frame moving with the beam centroid:

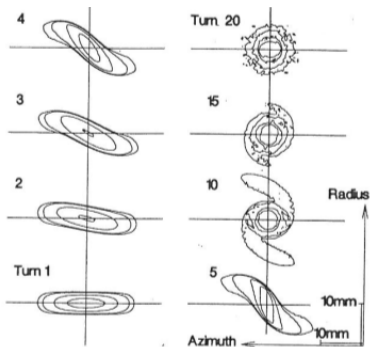


- Ignore acceleration phases to focus on beam dynamics due to space charges

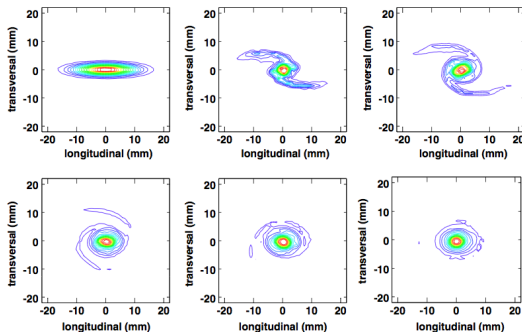
**BEAM PHENOMA WE WOULD LIKE TO BETTER  
UNDERSTAND**

# SPIRALING OF HIGH INTENSITY BEAMS

## PSI Injector II

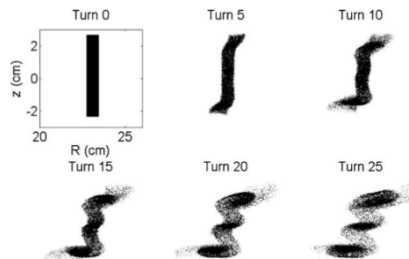
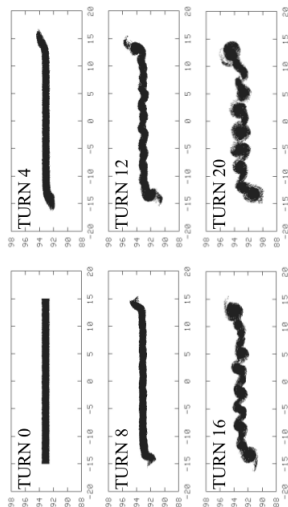


PICN (Adam)



OPAL-CYCL (Adelmann *et al.*)

# BREAKUP OF HIGH INTENSITY BEAMS

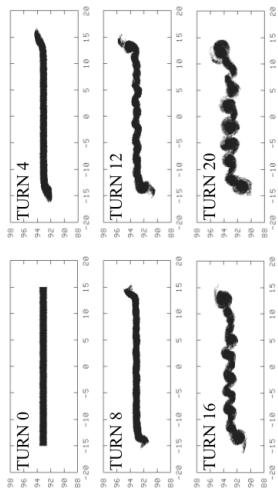
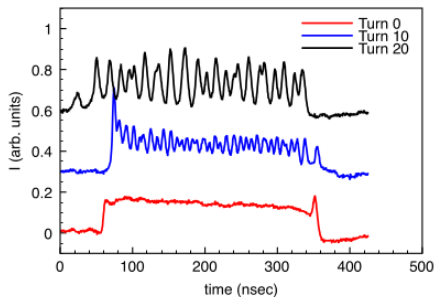


**CYCIAE 100**  
(Bi *et al.*)

**Small Isochronous Ring**  
(Pozdeyev *et al.*)

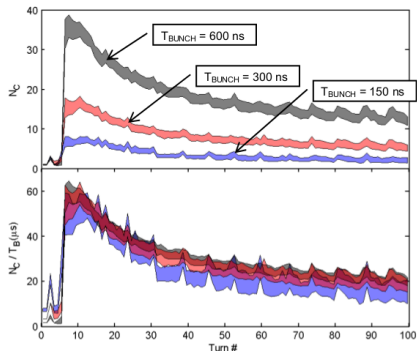
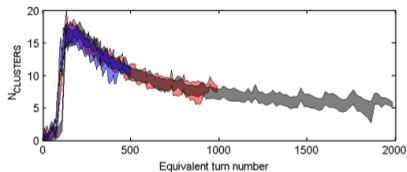
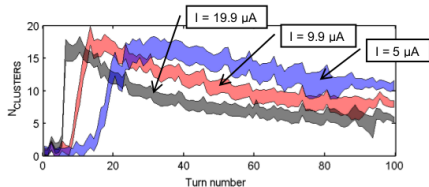
# BEAM BREAKUP

Detailed results from E. Pozdeyev, J. A. Rodriguez, F. Marti, and R. C. York, *Phys. Rev. STAB* **12**, 054202 (2009).



# BEAM BREAKUP

Detailed results from E. Pozdeyev, J. A. Rodriguez, F. Marti, and R. C. York, *Phys. Rev. STAB* **12**, 054202 (2009).



- ▶ Number of clusters independent of bunch density
- ▶ Number of clusters scales linearly with bunch length

## **DESCRIBING A NONNEUTRAL BEAM: A BRIEF SURVEY**

# METHOD I: SOLVING THE N-BODY PROBLEM

- ▶ An intuitive idea is to solve for the motion of all the particles iteratively
- ▶ At each time step  $i$ , solve

$$m \frac{d^2 \mathbf{x}_k^{(i)}}{dt^2} = q \left( \mathbf{E}(\mathbf{x}_k)^{(i-1)} + \frac{d\mathbf{x}_k^{(i)}}{dt} \times \mathbf{B}(\mathbf{x}_k) \right) \quad k = 1, \dots, N$$

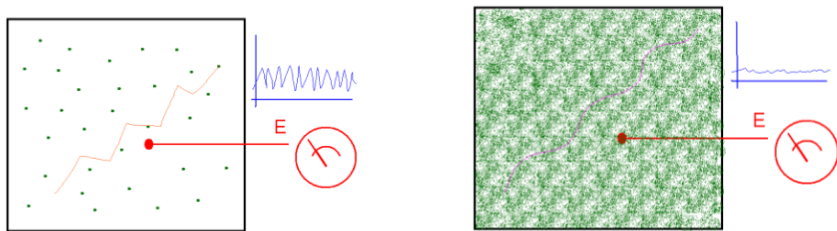
$$\mathbf{E}(\mathbf{x}_k)^{(i)} = \sum_{\substack{j=1 \\ j \neq k}}^N \frac{q_j}{4\pi\epsilon_0} \frac{\mathbf{x}_k^{(i)} - \mathbf{x}_j^{(i)}}{|\mathbf{x}_k^{(i)} - \mathbf{x}_j^{(i)}|^3}$$

- ▶ Use **Fast Multipole Method** to evaluate the electric field in  $O(N)$  operations<sup>1</sup>
- ▶ Still **not feasible computationally large  $N$**  – in the problems of interest  $N \sim 10^9 - 10^{15}$

---

<sup>1</sup>L. Greengard, V. Rokhlin, *J. Comput. Phys.* **73**, 325 (1987)

## METHOD II: COARSE-GRAIN AVERAGE IN PHASE SPACE



- For very large  $N$ , replace discrete particles with **smooth distribution function**  $f(\mathbf{x}, \mathbf{v}, t)$

$$f(\mathbf{x}, \mathbf{v}, t) d\mathbf{x} d\mathbf{v}$$

is the expected number of particles in the infinitesimal volume  $d\mathbf{x} d\mathbf{v}$ .

# THE PARTICLE IN CELL (PIC) APPROACH

- ▶ Sample the distribution function  $f$  with  $P$  “superparticles”  
 $f_p(\mathbf{x}, \mathbf{v}, t)$

$$f(\mathbf{x}, \mathbf{v}, t) = \sum_p f_p(\mathbf{x}, \mathbf{v}, t)$$

- ▶ Write  $f_p(\mathbf{x}, \mathbf{v}, t) = N_p S_x(\mathbf{x} - \mathbf{x}_p(t)) S_v(\mathbf{v} - \mathbf{v}_p(t))$   
 $S$ : shape functions for the “superparticle”  
 $N_p$ : number of physical particles in the “superparticle”
- ▶  $f_p$  evolves in time according to

$$\begin{aligned} \frac{dN_p}{dt} &= 0 & \frac{d\mathbf{x}_p}{dt} &= \mathbf{v}_p & \frac{d\mathbf{v}_p}{dt} &= \frac{q}{m} (-\nabla\phi_p + \mathbf{v}_p \times \mathbf{B}) \\ \nabla^2\phi_p &= -\frac{q}{\epsilon_0} \sum_p N_p S_x(\mathbf{x} - \mathbf{x}_p) \end{aligned}$$

- ▶ Difference with discrete particle simulations: the fields and trajectories are **smooth**

# THE PARTICLE IN CELL (PIC) APPROACH

- ▶ Advantages:

- ▶ Intuitive: Newton-Maxwell system as in the discrete particle case
- ▶ Easier to parallelize than continuum methods
- ▶ Works well for high-dimensional problems

- ▶ Disadvantages:

- ▶ Does not yield much theoretical insights
- ▶ Statistical noise

# MOMENT APPROACH

Vlasov equation: 
$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{q}{m} (-\nabla \phi + \mathbf{v} \times \mathbf{B}) \cdot \nabla_{\mathbf{v}} f = 0$$

- ▶ Taking the integrals  $\iiint d\mathbf{v}$ ,  $\iiint m\mathbf{v}d\mathbf{v}$  and  $\iiint mv^2/2d\mathbf{v}$  of this equation, we obtain the exact **fluid equations**:

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{V}) = 0 \quad \text{Continuity}$$

$$mn \left( \frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} \right) = en (-\nabla \phi + \mathbf{V} \times \mathbf{B}) - \nabla \cdot \mathbf{P} \quad \text{Momentum}$$

$$\frac{d}{dt} \left( \frac{3}{2} p \right) + \frac{5}{2} p \nabla \cdot \mathbf{U} + \boldsymbol{\pi} : \nabla \mathbf{U} + \nabla \cdot \mathbf{q} = 0 \quad (\text{Energy})$$

with  $\mathbf{p} = p\mathbf{I} + \boldsymbol{\pi}$ .

- ▶ **Closure problem:** for each moment, we introduce a new unknown  $\Rightarrow$  End up with too many unknowns
- ▶ Need **approximation** to close the moment hierarchy

## CLOSURE FOR TWO-DIMENSIONAL BEAM DYNAMICS

## IDENTIFYING SMALL PARAMETERS

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{q}{m} (-\nabla \phi + \mathbf{v} \times \mathbf{B}) \cdot \nabla_v f = 0$$

- ▶ Simplify geometry:  $\mathbf{B} = B_0 \mathbf{e}_z$  (uniform magnetic field)
- ▶ Normalize equation

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{1}{\epsilon} \left( -\frac{\delta^2}{\epsilon} \nabla \phi + \mathbf{v} \times \mathbf{e}_z \right) \cdot \nabla_v f = 0$$

$$\epsilon \equiv \frac{\rho}{a} \quad \delta^2 \equiv \frac{\omega_p^2}{\omega_c^2} = \frac{mn}{\epsilon_0 B^2}$$

- ▶  $a$ : characteristic size of the beam
- ▶  $\rho$ : Particle gyroradius in the moving frame  
(related to the magnitude of the beam mismatch oscillations)
- ▶ All cyclotrons satisfy  $\delta^2 \leq 1$ , and **most satisfy  $\delta^2 \ll 1$**

## DERIVING FLUID EQUATIONS FOR THE BEAM

- ▶ In general,  $\epsilon = \rho/a \lesssim 1$
- ▶ Focus on the regime  $\epsilon \sim \delta \ll 1$
- ▶ To lowest order in  $\delta$

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{1}{\epsilon} \left( -\frac{\delta^2}{\epsilon} \nabla \phi + \mathbf{v} \times \mathbf{e}_z \right) \cdot \nabla_{\mathbf{v}} f = 0$$

becomes

$$\therefore (\mathbf{v} \times \mathbf{e}_z) \cdot \nabla_{\mathbf{v}} f = 0 + O(\delta)$$

- ▶ The pressure tensor must then be **diagonal**<sup>2</sup> to lowest order in  $\delta$ :

$$\mathbf{P} = p_{\perp} \mathbf{I} + (p_{\parallel} - p_{\perp}) \mathbf{b} \mathbf{b} = \begin{pmatrix} p_{\perp} & 0 & 0 \\ 0 & p_{\perp} & 0 \\ 0 & 0 & p_{\parallel} \end{pmatrix} + O(\delta)$$

- ▶ Closed fluid equations can then be derived for the 2D dynamics in the plane perpendicular to the magnetic field<sup>2</sup>

---

<sup>2</sup>A.J. Cerfon *et al.*, *PRSTAB* **16**, 024202 (2013)

# MULTIPLE TIME SCALE ANALYSIS

- ▶  $\delta \ll 1$  means electrostatic force is much smaller than magnetic force
- ▶ Time scale for particle gyration much shorter than time scale for macroscopic evolution of the beam
- ▶ Situation lends itself to a multiple time scale analysis, in which the dynamics is averaged over the fast cyclotron time scale<sup>2</sup>
- ▶ One finds the following equation for the space charge dynamics:

$$\frac{\partial n}{\partial(\delta^2 t)} + \nabla \phi \times \mathbf{e}_z \cdot \nabla n = 0$$
$$\nabla^2 \phi = -n$$

- ▶ These equations describe the advection of the density profile in the velocity field  $\mathbf{E} \times \mathbf{B}/B^2$ , the so-called  $\mathbf{E} \times \mathbf{B}$  velocity
- ▶ Dynamics **independent of the value of  $\delta$**  except for the time scale of the observed phenomena
- ▶ Scaling **linear with  $\delta^2$ , i.e. density**, i.e. current (at fixed energy)

# ISOMORPHISM WITH 2D EULER EQUATIONS

## Beam vortex dynamics

$$\frac{\partial n}{\partial t} + \nabla \phi \times \mathbf{e}_z \cdot \nabla n = 0$$
$$\nabla^2 \phi = -n$$

$n$ : bunch density;  $\phi$ : electrostatic potential

## 2D incompressible Euler

$$\frac{\partial \omega}{\partial t} + \nabla \psi \times \mathbf{e}_z \cdot \nabla \omega = 0$$
$$\nabla^2 \psi = -\omega$$

$\omega$ : z-directed vorticity;  $\psi$ : stream function for the flow

- ▶ Isomorphism recognized a long time ago in a slightly different context<sup>3</sup>
- ▶ We proved that the isomorphism holds even for finite temperature beams
- ▶ We can use decades old fluid dynamics results to determine/understand the stability of bunch distributions

---

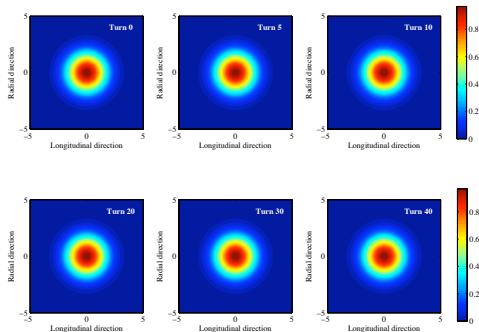
<sup>3</sup>C.F. Driscoll and K.S. Fine, *Phys.Fluids B* 2 1359 (1990)

## **EXPLAINING BEAM PHENOMENA**

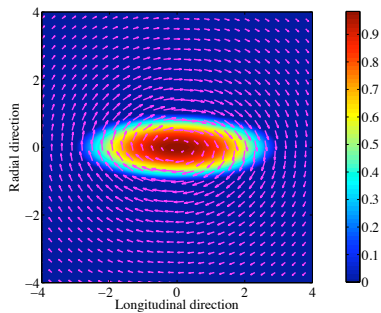
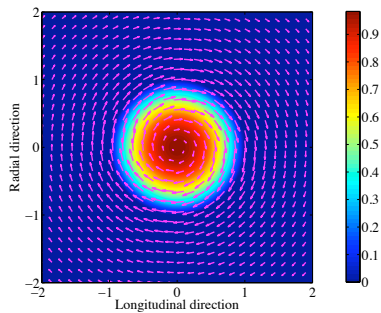
# STABILITY OF ROUND BEAMS

- ▶ Radial density distributions automatically satisfy the equations
- ▶ Well-known results from fluid theory of radially symmetric vortex patches:
  - ▶ If  $n(r)$  is **monotonically decreasing**, the bunch is **nonlinearly stable** to nonsymmetric density perturbations
  - ▶ Hollow density profiles can be unstable to these perturbations

Gaussian  $n(r)$ ,  $\delta^2 = 0.8$

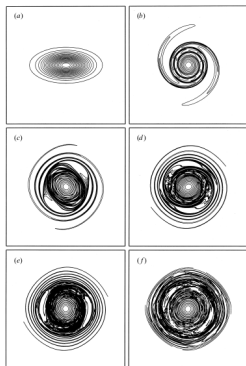


# ExB ADVECTION



# ELLIPTIC BUNCHES WITH SMOOTH DENSITY PROFILE

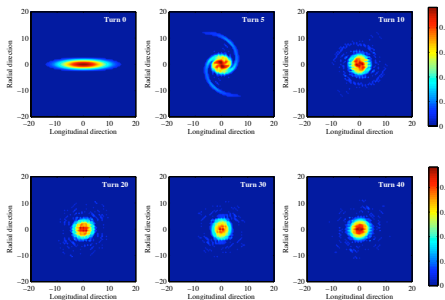
- ▶ More complicated case. Answer depends on the smoothness of the profile
- ▶ For reasonably smooth profile, “**axisymmetrization principle**”<sup>4</sup>
- ▶ Also known as “inviscid damping”, Euler analog of Landau damping



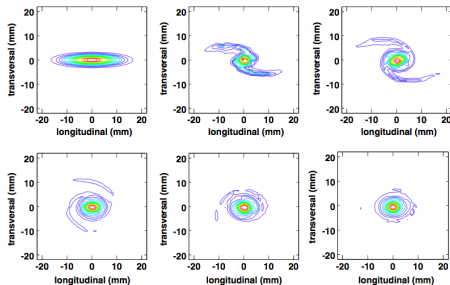
<sup>4</sup>M.V. Melander, J.C. McWilliams and N.J. Zabusky, J. Fluid Mech. 178 (1987) 137

# BEAM SPIRALING A.K.A AXISYMMETRIZATION

Our simulation



3D PIC (OPAL) simulation<sup>a</sup>

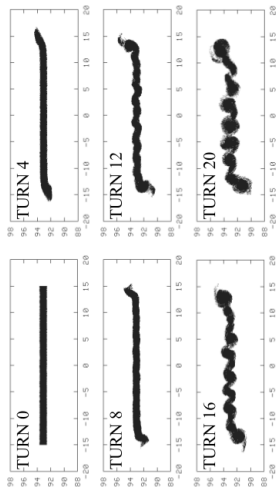
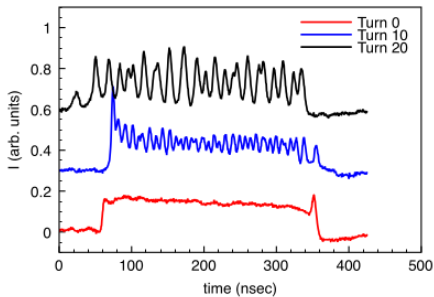


<sup>a</sup>J. Yang *et al.*, *Phys. Rev. ST AB* 13 064201

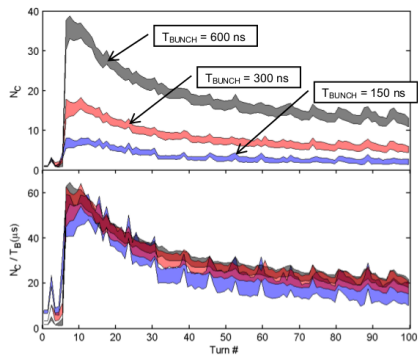
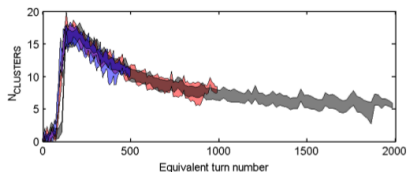
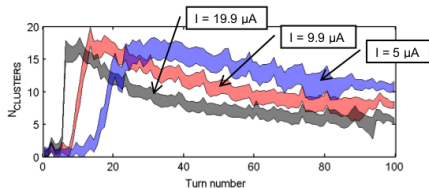
- In PSI Injector II, formation of a round stable core after 40 turns (good)
- In machines with lower  $\delta^2$ , core and halo take longer to form  
Potentially bad situation if low density halo forms with high energy

# BEAM BREAKUP

Results from E. Pozdeyev, J. A. Rodriguez, F. Marti, and R. C. York,  
*Phys. Rev. STAB* **12**, 054202 (2009).



# BEAM BREAKUP



- ▶ Number of clusters **independent of bunch density**  
⇒ Automatic in our model
- ▶ Number of clusters **scales linearly with bunch length**  
⇒ Consequence of shear inviscid instability

# SHEAR INVISCID INSTABILITY

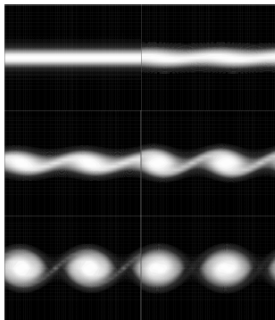
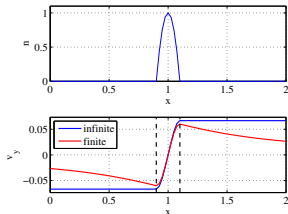
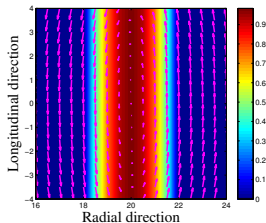
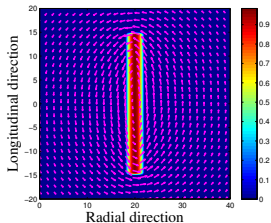


Figure 5. Contour plots of spanwise vorticity. Linear and non-linear two-dimensional evolution of a disturbance for  $M_\infty = 0.4$ ,  $Re = 500$  and  $\alpha \approx 0.82$ . The disturbance here was composed of only a fundamental mode. The frames presented correspond to the non-dimensional times 10, 55, 60, 65, 85 and 160. Relative to the parameter  $\omega t$ , the frames correspond to 8.2, 45.1, 49.2, 53.3, 69.7 and 131.2.

Figure: Linear and nonlinear instability of a compressible shear layer described by the Navier Stokes equation<sup>5</sup>

<sup>5</sup>R.A. Coppola Germanos, L. Franco de Souza, M.A. Faraco de Medeiros, *J. Braz. Soc. Mech. Sci & Eng.* **31**, 125 (2009)

# SHEAR INVISCID INSTABILITY



- For simplicity, assume infinite bunch in the  $y$  direction
- Use modal approach and solve Rayleigh eigenvalue equation<sup>a</sup>

---

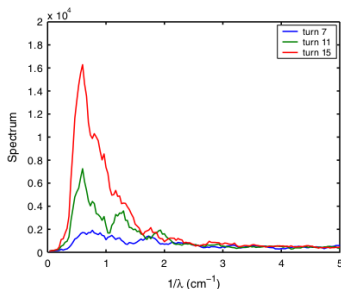
<sup>a</sup>P.G. Drazin and W.H. Reid, *Hydrodynamic Stability*, Cambridge University Press (1981)

# SHEAR INVISCID INSTABILITY

- Rayleigh's eigenvalue equation for a shear layer:

$$(V_y - \omega) \left( \frac{d^2 \hat{\phi}}{dx^2} - k^2 \phi \right) - \frac{d^2 V_y}{dx^2} \hat{\phi} = 0$$

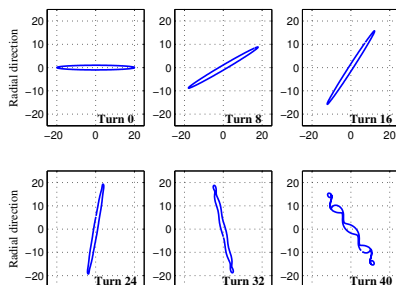
- Solving this equation numerically, we find that the wave number  $k$  with the largest growth rate is such that  $k \approx 0.8 \text{ cm}^{-1}$
- Agrees well with PIC observations for SIR



# ELLIPTIC BUNCHES WITH UNIFORM DENSITY

- ▶ Classical case in fluid dynamics: **uniform density profile**  
Call  $a$  the semi-major axis and  $b$  the semi-minor axis
- ▶ If  $a/b < 3$ , bunch is linearly and nonlinearly **stable** to edge perturbations  
If  $a/b > 3$ , bunch is linearly and nonlinearly **unstable** to edge perturbations
- ▶ Instability is a potential mechanism for **beam breakup**

Uniform  $n$ ,  $\delta^2 = 0.2$ ,  $a/b = 20$



## **ONGOING WORK: GENERALIZING OUR MODEL**

## ARBITRARY TEMPERATURE / INITIAL CONDITIONS

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{1}{\epsilon} \left( -\frac{\delta^2}{\epsilon} \nabla \phi + \mathbf{v} \times \mathbf{e}_z \right) \cdot \nabla_{\mathbf{v}} f = 0$$

- We treated the case  $\epsilon \ll 1$ ; what happens when  $\epsilon \sim 1$ ?

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{1}{\epsilon} \mathbf{v} \times \mathbf{e}_z \cdot \nabla_{\mathbf{v}} f = 0$$

- To lowest order,  $f = f(x, y, v_{\perp}, \varphi) \Rightarrow f$  is **not independent of gyrophase**  $\Rightarrow$  In principle, need to resolve betatron time scale
- Idea: (called gyrokinetics) **Solve for distribution of gyrocenters**  $\bar{f}(\mathbf{R}, v_{\perp}, \varphi)$  where  $\mathbf{R} = \mathbf{r} + \mathbf{v} \times \mathbf{e}_z / \omega_c$
- After transformation  $(\mathbf{r}, v_{\perp}, \varphi) \rightarrow (\mathbf{R}, \mu, \varphi)$ , one finds

$$\frac{\partial \bar{f}}{\partial \varphi} = 0 + O(\delta)$$

# GYROKINETICS FOR CYCLOTRON BEAMS

$$\frac{\partial \bar{f}}{\partial t} + \frac{q}{m\omega_c} \mathbf{b} \times \langle \nabla \phi \rangle_\varphi \cdot \nabla_{\mathbf{R}} \bar{f} = 0$$
$$\nabla^2 \phi(\mathbf{r}, t) = \frac{\omega_c}{\epsilon_0} \iint \bar{f}(\mathbf{r} + \frac{\mathbf{v}_\perp \times \mathbf{b}}{\omega_c}, \mu, t) d\mu d\varphi$$

- ▶ Problem is now 3D and kinetic  $\Rightarrow$  slightly more complicated
- ▶ Thanks to gyroaveraging, **only solve on space charge time scale**
- ▶ A solver is being developed by NYU graduate student J. Guadagni, and is almost fully written
- ▶ First results will be presented at APS-DPP meeting at the end of October
- ▶ Next step: **include spatial inhomogeneities for the magnetic field**